

# ON THE SUBSET COMBINATORICS OF $G$ -SPACES

IGOR PROTASOV AND SERGII SLOBODIANIUK

**ABSTRACT.** Let  $G$  be a group and let  $X$  be a transitive  $G$ -space. We classify the subsets of  $X$  with respect to a translation invariant ideal  $J$  in the Boolean algebra of all subsets of  $X$ , introduce and apply the relative combinatorial derivations of subsets of  $X$ . Using the standard action of  $G$  on the Stone-Ćech compactification  $\beta X$  of the discrete space  $X$ , we characterize the points  $p \in \beta X$  isolated in  $Gp$  and describe a size of a subset of  $X$  in terms of its ultracompanions in  $\beta X$ . We introduce and characterize scattered and sparse subsets of  $X$  from different points of view.

## 1. INTRODUCTION

Let  $G$  be a group and let  $X$  be a transitive  $G$ -space with the action  $G \times X \rightarrow X$ ,  $(g, x) \mapsto gx$ . If  $X = G$  and  $gx$  is a product of  $g$  and  $x$  then  $X$  is called the *left regular  $G$ -space*.

A family  $J$  of subsets of  $X$  is called an ideal in the Boolean algebra  $\mathcal{P}_X$  of all subsets of  $X$  if  $X \notin J$  and  $A, B \in J$ ,  $C \subset A$  imply  $A \cup B \in J$  and  $C \in J$ . The ideal of all finite subsets of  $X$  is denoted by  $[X]^{<\omega}$ . An ideal  $J$  is *translation invariant* if  $gA \in J$  for all  $g \in G$ ,  $A \in J$ , where  $gA = \{ga : a \in A\}$ . If  $X$  is finite then  $J = \{\emptyset\}$  so in what follows all  $G$ -spaces are supposed to be infinite.

Now we fix a translation invariant ideal  $J$  in  $\mathcal{P}_X$  and say that a subset  $A$  of  $X$  is

- *$J$ -large* if  $X = FA \cup I$  for some  $F \in [G]^{<\omega}$  and  $I \in J$ ;
- *$J$ -small* if  $L \setminus A$  is  $J$ -large for every  $J$ -large subset  $L$  of  $X$ ;
- *$J$ -thick* if  $\text{Int}_F(A) \notin J$  for each  $F \in [G]^{<\omega}$ , where  $\text{Int}_F(A) = \{a \in A : Fa \subseteq A\}$ ;
- *$J$ -prethick* if  $FA$  is thick for some  $F \in [G]^{<\omega}$ .

If  $J = \emptyset$  we omit the prefix  $J$  and get a well-known classification of subsets of a  $G$ -spaces by their combinatorial size (see the survey [11]).

In the case of the left regular  $G$ -spaces, the notions of  $J$ -large and  $J$ -small subsets appeared in [1].

We say that a mapping  $\Delta_J : \mathcal{P}_X \rightarrow \mathcal{P}_G$  defined by

$$\Delta_J(A) = \{g \in G : gA \cap A \notin J\}$$

is a *combinatorial derivation relatively to the ideal  $J$* . If  $X$  is the left regular  $G$ -space and  $J = [X]^{<\omega}$ , the mapping  $\Delta_J$  was introduced in [12] under the name combinatorial derivation and studied in [13].

In Section 2 we prove that if a subset  $A$  of  $X$  is not  $J$ -small then  $\Delta_J(A)$  is large in  $G$ . For the left regular  $G$ -space  $X$  and  $J = [X]^{<\omega}$ , this statement was proved in [6].

We endow  $X$  with the discrete topology and take the points of  $\beta X$ , the Stone-Ćech compactification of  $X$ , to be the ultrafilters on  $X$ , with the points of  $X$  identified with the principal

---

1991 *Mathematics Subject Classification.* 20F69, 22A15, 54D35.

*Key words and phrases.*  $G$ -space, relative combinatorial derivation, Stone-Ćech compactification, ultracompanion, sparse and scattered subsets.

ultrafilters on  $X$ . The topology on  $\beta X$  can be defined by stating that the set of the form  $\overline{A} = \{p \in \beta X : A \in p\}$ , where  $A$  is a subset of  $X$ , form a base for the open sets. We note the sets of this form are clopen and that for any  $p \in \beta X$  and  $A \subset X$ ,  $A \in p$  if and only if  $p \in \overline{A}$ . We denote  $A^* = \overline{A} \cap X^*$ , where  $X^* = \beta X \setminus X$ . The universal property of  $\beta X$  states that every mapping  $f : X \rightarrow Y$ , where  $Y$  is a compact Hausdorff space, can be extended to the continuous mapping  $f^\beta : \beta X \rightarrow Y$ .

Now we endow  $G$  with the discrete topology and, using the universal property of  $\beta G$ , extend the group multiplication from  $G$  to  $\beta G$  (see [8, Chapter 4]), so  $\beta G$  becomes a compact right topological semigroup.

We define the action of  $\beta G$  on  $\beta X$  in two steps. Given  $g \in G$ , the mapping

$$x \mapsto gx : X \rightarrow \beta X$$

extends to the continuous mapping

$$p \mapsto gp : \beta X \rightarrow \beta X.$$

Then, for each  $p \in \beta X$ , we extend the mapping  $g \mapsto gp : G \rightarrow \beta X$  to the continuous mapping

$$q \mapsto qp : \beta G \rightarrow \beta X.$$

Let  $q \in \beta G$  and  $p \in \beta X$ . To describe a base for the ultrafilter  $qp \in \beta X$ , we take any element  $Q \in q$  and, for every  $g \in Q$  choose some element  $P_x \in p$ . Then  $\bigcup_{g \in Q} gP_x \in qp$ , and the family of subsets of this form is a base for the ultrafilter  $qp$ .

Given a subset  $A$  of  $X$  and an ultrafilter  $p \in X^*$  we define a  $p$ -companion of  $A$  by

$$\Delta_p(A) = A^* \cap Gp = \{gp : g \in G, A \in gp\},$$

and say that a subset  $S$  of  $X^*$  is an *ultracompanion* of  $A$  if  $S = \Delta_p(A)$  for some  $p \in X^*$ .

In Section 3 we characterize the subsets of  $X$  of different types in terms of their ultracompanions. For example a subset  $A$  of  $X$  is  $J$ -large if and only if  $\Delta_p(A) \neq \emptyset$  for each  $p \in \check{J}$ , where  $\check{J} = \{p \in X^* : X \setminus I \in p \text{ for every } I \in J\}$ . For the left regular  $X$  and  $J = \{\emptyset\}$ , these characterizations are obtained in [15].

In Section 4 we describe the points  $p \in \beta X$  isolated in  $Gp$  and introduce the piecewise shifted  $FP$ -sets in  $X$  to characterize the subsets  $A \subseteq X$  such that  $\Delta_p(A)$  is discrete for each  $p \in X^*$ .

In Section 5 we extend the notions scattered and sparse subsets from groups [3] to  $G$ -space and characterize these subsets from different points of view.

## 2. RELATIVE COMBINATORIAL DERIVATIONS

Let  $X$  be a transitive  $G$ -space and let  $J$  be a translation invariant ideal in  $\mathcal{P}_X$ .

**Lemma 2.1.** *For a subset  $A$  of  $X$ , the following statements are equivalent*

- (i)  $A$  is  $J$ -small;
- (ii)  $G \setminus FA$  is  $J$ -large for each  $F \in [G]^{<\omega}$ ;
- (iii)  $A$  is not  $J$ -prethick.

*Proof.* Apply the arguments proving Theorem 2.1 in [1]. □

The next lemma follows directly from the definition of  $J$ -small subsets.

**Lemma 2.2.** *The family of all  $J$ -small subsets of  $X$  is a translation invariant ideal in  $\mathcal{P}_X$ .*

**Lemma 2.3.** *Let  $L$  be a  $J$ -large subset of  $X$ . Then given a partition  $L = A \cup B$ , either  $\Delta_J(A)$  is large or  $B$  is  $J$ -large.*

*Proof.* We take  $F \in [G]^{<\omega}$  and  $I \in J$  such that  $G = F(A \cup B) \cup I$ . Assume that  $G \neq F\Delta_J(A)$  and show that  $B$  is  $J$ -large.

Let  $F = \{f_1, \dots, f_k\}$ . We take  $g \in G \setminus F\Delta_J(A)$  and put  $I_i = f_i^{-1}gA \cap A$ ,  $i \in \{1, \dots, k\}$ . Since  $g \notin f_i\Delta_J(A)$ , we have  $I_i \in J$  and  $f_i^{-1}gx \notin A$  for each  $x \in A \setminus I_i$ .

If  $x \in X$  and  $F^{-1}gx \cap L = \emptyset$  then  $gx \notin FL$  so  $gx \in I$  and  $x \in g^{-1}I$ . We put

$$I' = I_1 \cup \dots \cup I_k \cup g^{-1}I.$$

If  $x \in A \setminus I'$  then there is  $i \in \{1, \dots, k\}$  such that  $f_i^{-1}gx \in A \cup B$ . Since  $f_i^{-1}gx \notin A$ , we have  $f_i^{-1}gx \in B$ . Hence,  $A \setminus I' \subseteq F^{-1}gB$  and

$$G = F(A \setminus I') \cup FI' \cup FB \cup I = FF^{-1}gB \cup FB \cup (FI' \cup I),$$

and we conclude that  $B$  is  $J$ -large.  $\square$

**Theorem 2.4.** *If a subset  $A$  of  $X$  is  $J$ -prethick then  $\Delta_J(A)$  is large.*

*Proof.* By Lemma 2.1,  $A$  is not  $J$ -small. We take a  $J$ -large subset  $L$  such that  $L \setminus A$  is not  $J$ -large. Since  $L = (L \cap A) \cup (L \setminus A)$ , by Lemma 2.3,  $\Delta_J(L \cap A)$  is large so  $\Delta_J(A)$  is large.  $\square$

**Corollary 2.5.** *If an  $J$ -prethick subset  $A$  of  $X$  is finitely partitioned  $A = A_1 \cup \dots \cup A_n$  then  $\Delta_J(A_i)$  is large for some  $i \in \{1, \dots, n\}$*

*Proof.* By Lemma 2.2 some cell  $A_i$  is prethick. Apply Theorem 2.4.  $\square$

**Remark 2.6.** Given a translation invariant ideal  $J$  in  $\mathcal{P}_X$ , there is a function  $\Phi_J : \mathbb{N} \rightarrow \mathbb{N}$  such that, for any  $n$ -partition  $X_1 \cup \dots \cup X_n$  of  $X$ , there exists  $A_i$  and  $F \in [G]^{<\omega}$  such that  $G = F\Delta_J(A_i)$  and  $|F| \leq \Phi_J(n)$ . These functions are intensively studied in [2] and [4].

### 3. ULTRACOMPANIONS

Given a translation invariant ideal  $J$  in  $\mathcal{P}_X$ , we denote

$$\check{J} = \{p \in X^* : X \setminus I \in p \text{ for each } I \in J\},$$

and observe that  $\check{J}$  is closed in  $X^*$  and  $gp \in \check{J}$  for all  $g \in G$  and  $p \in \check{J}$ .

**Theorem 3.1.** *For a subset  $A$  of  $X$ , the following statements hold*

- (i)  $A$  is  $J$ -large if and only if  $\Delta_p(A) \neq \emptyset$  for each  $p \in \check{J}$ ;
- (ii)  $A$  is  $J$ -thick if and only if there exists  $p \in \check{J}$  such that  $\Delta_p(A) = Gp$ ;
- (iii)  $A$  is  $J$ -prethick if and only if there exists  $p \in \check{J}$  and  $F \in [G]^{<\omega}$  such that  $\Delta_p(FA) = Gp$ ;
- (iv)  $A$  is  $J$ -small if and only if for every  $p \in \check{J}$  and every  $F \in [G]^{<\omega}$ , we have  $\Delta_p(A) \neq Gp$ .

*Proof.* (i) Suppose that  $A$  is  $J$ -large and choose  $F \in [G]^{<\omega}$  and  $I \in J$  such that  $X = FA \cup I$ . We take an arbitrary  $p \in \check{J}$  and choose  $g \in F$  such that  $gA \in p$  so  $A \in g^{-1}p$  and  $\Delta_p(A) \neq \emptyset$ .

Assume that  $\Delta_p(A) \neq \emptyset$  for each  $p \in J$ . Given  $p \in J$ , we choose  $g_p \in G$  such that  $A \in g_p p$ . Then we consider a covering of  $\check{J}$  by the subsets  $\{g_p^{-1}A^* : p \in \check{J}\}$  and choose its finite subcovering  $g_{p_1}^{-1}A^*, \dots, g_{p_n}^{-1}A^*$ . We take  $I \in J$  and  $H \in [X]^{<\omega}$  such that  $X \setminus (g_{p_1}^{-1}A^* \cup \dots \cup g_{p_n}^{-1}A^*) = I \cup H$ . At last, we choose  $F \in [G]^{<\omega}$  such that  $\{g_{p_1}^{-1}, \dots, g_{p_n}^{-1}\} \subseteq F$  and  $H \subseteq FA$ . Then  $X = FA \cup I$  and  $A$  is  $J$ -large.

- (ii) We note that  $A$  is  $J$ -thick if and only if  $X \setminus A$  is not  $J$ -large and apply (i).
- (iii) follows from (ii).
- (iv) follows from (iii) and Lemma 2.1. □

We suppose that  $J \neq \{\emptyset\}$  and say that a subset  $A$  of  $X$  is  $J$ -thin if, for every  $F \in [G]^{<\omega}$ , there exists  $I \in J$  such that  $|Fa \cap A| \leq 1$  for each  $a \in A \setminus I$ .

**Theorem 3.2.** *A subset  $A$  of  $X$  is  $I$ -thin if and only if  $\Delta_p(A) \leq 1$  for each  $p \in J$ .*

*Proof.* Suppose that  $A$  is not  $J$ -thin and choose  $F \in [G]^{<\omega}$  such that, for each  $I \in J$ , there is  $a_I \in A \setminus I$  satisfying  $Fa_I \cap A \neq \{a_I\}$ . We pick  $g_I \in F$  and  $b_I \in A$  such that  $g_I a_I = b_I$  and  $b_I \in A$ . Then we put  $A_I = \{a_{I'} : I \subseteq I', I' \in J\}$  and take  $p \in \check{J}$  such that  $A_I \in p$  for each  $I \in J$ . Since  $p$  is an ultrafilter, there exists  $g \in F$  such that  $gp \neq p$  and  $A \in gp$ . Hence  $\{p, gp\} \subseteq \Delta_p(A)$  and  $|\Delta_p(A)| > 1$ .

Assume that  $|\Delta_p(A)| > 1$  for some  $p \in J$ . We pick distinct  $g_1 p, g_2 p \in \Delta_p(A)$  and put  $F = \{g_2 g_1^{-1}\}$ . Since  $A \setminus I \in g_1 p \cap g_2 p$  for each  $I \in J$ , there is  $a_I \in A \setminus I$  such that  $g_2^{-1} g_1 a_I \in A \setminus \{a_I\}$ . Hence,  $A$  is not  $J$ -thin. □

**Remark 3.3.** We say that a non-empty subset  $S$  of  $\beta X^*$  is invariant if  $gS \subseteq S$  for each  $g \in G$ . It is easy to see that each closed invariant subset  $S$  of  $X$  contains a minimal by inclusion closed invariant subset  $M$  and  $M = cl(Gp)$  for each  $p \in M$ . By analogy with Theorem 4.39 from [8], we can prove that for  $p \in X^*$  the subset  $cl(Gp)$  is minimal if and only if, for every  $P \in p$ , there exists  $F \in [G]^\omega$  such that  $Gp \subseteq (FP)^*$ .

**Remark 3.4.** Given a translation invariant ideal  $J$  in  $\mathcal{P}_X$ , we denote

$$K(\check{J}) = \bigcup \{M : M \text{ is a minimal closed invariant subset of } \check{J}\}.$$

By analogy with Theorem 4.40 from [8], we can prove that  $p \in cl(K(\check{J}))$  if and only if each subset  $P \in p$  is  $J$ -prethick. It is worth to be mentioned that each closed invariant subset  $S$  of  $X^*$  is of the form  $S = \check{J}$  for some translation invariant ideal  $J$  in  $\mathcal{P}_X$ .

**Remark 3.5.** By Theorem 6.30 from [8], for every infinite group of cardinality  $\aleph$ , there exists  $2^{2^\aleph}$  distinct minimal closed invariant subsets of  $G^*$ . We show that this statement fails to be true for  $G$ -spaces. Let  $X = \omega$  and  $G$  be the group of all permutations of  $X$ . If  $S$  is a closed invariant subset of  $X^*$  then  $S = X^*$ .

**Remark 3.6.** We describe a relationship between ultracompanions and relative combinatorial derivations. Let  $J$  be a translation invariant ideal in  $\mathcal{P}_X$ ,  $A \subseteq X$ ,  $p \in \check{J}$ . We denote  $A_p = \{g \in G : A \in gp\}$  so  $\Delta_p(A) = A_p p$ . Then

$$\Delta_J(A) = \bigcap \{A_p^{-1} : p \in \check{J}, A \in p\}.$$

#### 4. ISOLATED POINTS

Given any  $p \in X^*$ , we put

$$St(p) = \{g \in G : gp = p\},$$

and note that, by [8, Lemma 3.33],  $gp = p$  if and only if there exists  $P \in p$  such that  $gx = x$  for each  $x \in P$ .

**Theorem 4.1.** *For every  $p \in X^*$ , the following statements are equivalent*

- (i)  $p$  is not isolated in  $Gp$ ;
- (ii) there exists  $q \in (G \setminus St(p))^*$  such that  $qp = p$ ;
- (iii) there exists  $\varepsilon \in (G \setminus St(p))^*$  such that  $\varepsilon\varepsilon = \varepsilon$  and  $\varepsilon p = p$ .

*Proof.* The implications (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (i) are evident.

(ii)  $\Rightarrow$  (iii). In view of Theorem 2.5 from [8], it suffices to show that the set

$$S = \{q \in (G \setminus St(p))^* : qp = p\}$$

is a subsemigroup of  $G^*$ . Let  $q, r \in S$ ,  $Q \in q$ . For each  $x \in Q$ , we choose  $R_x \in r$  such that  $x^{-1}St(p) \cap R_x = \emptyset$ . Then  $xy \notin St(p)$  for each  $y \in R_x$ . We put

$$P = \bigcup_{x \in Q} xR_x,$$

and note that  $P \in qr$  and  $P \cap St(p) = \emptyset$ . Hence  $qr \in S$ .  $\square$

**Remark 4.2.** For each  $g \in G$ , the mapping  $p \mapsto gp : \beta X \rightarrow \beta X$  is a homeomorphism. It follows that  $Gp$  has an isolated point if and only if  $Gp$  is discrete.

Let  $(g_n)_{n \in \omega}$  be sequence in  $G$  and let  $(x_n)_{n \in \omega}$  be a sequence in  $X$  such that

- (1)  $\{g_0^{\varepsilon_0} \dots g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0, 1\}\} \cap \{g_0^{\varepsilon_0} \dots g_m^{\varepsilon_m} x_m : \varepsilon_i \in \{0, 1\}\} = \emptyset$  for all distinct  $m, n \in \omega$ ;
- (2)  $|\{g_0^{\varepsilon_0} \dots g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0, 1\}\}| = 2^{n+1}$  for every  $n \in \omega$ .

We say that a subset  $Y$  of  $X$  is a *piecewise shifted FP-set* if there exist  $(g_n)_{n \in \omega}$ ,  $(x_n)_{n \in \omega}$  satisfying (1) and (2) such that

$$Y = \{g_0^{\varepsilon_0} \dots g_n^{\varepsilon_n} x_n : \varepsilon_i \in \{0, 1\}, n \in \omega\}.$$

For definition of an *FP-set* in a group see [8, p. 108].

**Theorem 4.3.** *Let  $p$  be an ultrafilter from  $X^*$  such that  $Gp$  is not discrete. Then every subset  $P \in p$  contains a piecewise shifted FP-set.*

*Proof.* We choose  $g_0 \in G$  such that  $p \neq g_0 p$ ,  $P \in g_0 p$  and take  $P_0 \subseteq P$ ,  $P_0 \in p$  such that  $g_0 P_0 \cap P_0 = \emptyset$ . We pick an arbitrary  $x_0 \in P_0$ .

Suppose that the elements  $g_0, \dots, g_n$  from  $G$  and  $x_0, \dots, x_n$  from  $X$  have been chosen so that

- (3)  $g_0^{\varepsilon_0} \dots g_k^{\varepsilon_k} x_k \in P$  for all  $\varepsilon_i \in \{0, 1\}$  and  $k \leq n$ ;
- (4)  $\{g_0^{\varepsilon_0} \dots g_k^{\varepsilon_k} x_k : \varepsilon_i \in \{0, 1\}\} \cap \{g_0^{\varepsilon_0} \dots g_m^{\varepsilon_m} x_m : \varepsilon_i \in \{0, 1\}\} = \emptyset$  for all  $k < m \leq n$ ;
- (5)  $|\{g_0^{\varepsilon_0} \dots g_k^{\varepsilon_k} x_k : \varepsilon_i \in \{0, 1\}\}| = 2^{k+1}$  for all  $k \leq n$ ;
- (6)  $P \in g_0^{\varepsilon_0} \dots g_k^{\varepsilon_k} p$  for all  $\varepsilon_i \in \{0, 1\}$  and  $k \leq n$ ;
- (7)  $|\{g_0^{\varepsilon_0} \dots g_k^{\varepsilon_k} p : \varepsilon_i \in \{0, 1\}\}| = 2^{k+1}$  for all  $k \leq n$ .

Since  $p$  is not isolated in  $Gp$ , we use (6) and (7) to choose  $g_{n+1} \in G$  such that  $P \in g_0^{\varepsilon_0} \dots g_{n+1}^{\varepsilon_{n+1}} p$  for all  $\varepsilon_i \in \{0, 1\}$  and  $|\{g_0^{\varepsilon_0} \dots g_{n+1}^{\varepsilon_{n+1}} p : \varepsilon_i \in \{0, 1\}\}| = 2^{n+2}$ .

Then we choose  $P_{n+1} \in p$  such that  $g_0^{\varepsilon_0} \dots g_{n+1}^{\varepsilon_{n+1}} P_{n+1} \subseteq P$  for all  $\varepsilon_i \in \{0, 1\}$  and

$$g_0^{\varepsilon_0} \dots g_{n+1}^{\varepsilon_{n+1}} P_{n+1} \cap g_0^{\delta_0} \dots g_{n+1}^{\delta_{n+1}} P_{n+1} = \emptyset$$

for all distinct  $(\varepsilon_0, \dots, \varepsilon_{n+1})$  and  $(\delta_0, \dots, \delta_{n+1})$  from  $\{0, 1\}^{n+2}$

We pick  $x_{n+1} \in P_{n+1}$  so that

$$\{g_0^{\varepsilon_0} \dots g_{n+1}^{\varepsilon_{n+1}} x_{n+1} : \varepsilon_i \in \{0, 1\}\} \cap \{g_0^{\varepsilon_0} \dots g_k^{\varepsilon_k} x_k : \varepsilon_i \in \{0, 1\}\} = \emptyset$$

for each  $k \leq n$ .

After  $\omega$  steps, we get the sequences  $(g_n)_{n \in \omega}$  and  $(x_n)_{n \in \omega}$  which define the desired  $FP$ -set in  $P$ .  $\square$

**Theorem 4.4.** *For an infinite subset  $A$  of a  $G$ -space  $X$ , the following statements are equivalent*

- (i)  $Gp$  is discrete for each  $p \in A^*$ ;
- (ii)  $A$  contains no piecewise shifted  $FP$ -sets.

*Proof.* The implication (ii)  $\Rightarrow$  (i) follows from Theorem 4.3. To prove (i)  $\Rightarrow$  (ii), we suppose that  $A$  contains a piecewise shifted  $FP$ -set  $Y$  defined by the sequence  $(g_n)_{n \in \omega}$  and  $(x_n)_{n \in \omega}$ . By [8, Theorem 5.12], there is an idempotent  $\varepsilon \in G^*$  such that, for each  $m \in \omega$ ,

$$\{g_m^{\varepsilon_m} \dots g_n^{\varepsilon_n} : \varepsilon_i \in \{0, 1\}, m < n < \omega\} \in \varepsilon.$$

We take an arbitrary  $q \in A^*$  such that  $\{x_n : n \in \omega\} \in q$ . Put  $p = \varepsilon q$ . Since  $Y \subseteq A$ , we have  $p \in A^*$ . Clearly,  $\varepsilon p = p$ . We note that  $g_m^{\varepsilon_m} \dots g_n^{\varepsilon_n} \in St(p)$  if and only if  $\varepsilon_m = \dots = \varepsilon_n = 0$ . Hence  $G \setminus St(p) \in \varepsilon$  and, applying Theorem 4.1, we conclude that  $p$  is not isolated in  $Gp$ .  $\square$

## 5. SCATTERED AND SPARSE SUBSETS OF $G$ -SPACES

Given  $F \in [G]^{<\omega}$  and  $x \in X$ , we denote  $B(x, F) = Fx \cup \{x\}$  and say that  $B(x, F)$  is a *ball of radius  $F$  around  $x$* . For subset  $Y$  of  $X$  and  $y \in Y$ , we denote  $B_Y(y, F) = B(y, F) \cap Y$ .

A subset  $A$  of  $X$  is called

- *scattered* if, for every infinite subset  $Y$  of  $X$ , there exists  $H \in [G]^{<\omega}$  such that, for every  $F \in [G]^{<\omega}$  there is  $y \in Y$  such that  $B_Y(y, F) \cap B_Y(y, H) = \emptyset$ ;
- *sparse* if, for every infinite subset  $Y$  of  $X$ , there exists  $H \in [G]^{<\omega}$  such that, for every  $F \in [G]^{<\omega}$  there is  $y \in Y$  such that  $B_A(y, F) \cap B_A(y, H) = \emptyset$ .

Clearly, each sparse subset is scattered. The sparse subsets of groups were introduced in [7] and studied in [9] [10]. From the asymptotic point of view [16], the scattered subsets of  $G$ -spaces can be considered as counterparts of the scattered subspaces of topological spaces.

**Proposition 5.1.** *A subset  $A$  of a  $G$ -space  $X$  is sparse if and only if  $\Delta_p(A)$  is finite for each  $p \in X^*$ .*

*Proof.* Repeat the arguments proving Theorem 10 in [14].  $\square$

**Proposition 5.2.** *A subset  $A$  of a  $G$ -space  $X$  is scattered if and only if, for every infinite subset  $Y$  of  $X$ , there exists  $p \in Y^*$  such that  $\Delta_p(Y)$  is finite.*

*Proof.* Repeat the arguments proving Proposition 1 in [3].  $\square$

To formulate further results, we need some asymptology (see [16, Chapter 1]). Let  $G_1, G_2$  be groups,  $X_1$  be a  $G_1$ -space,  $X_2$  be a  $G_2$ -space,  $Y_1 \subseteq X_1, Y_2 \subseteq X_2$ . A mapping  $f : Y_1 \rightarrow Y_2$  is called a  $\prec$ -mapping if, for every  $F \in [G_1]^{<\omega}$ , there exists  $H \in [G_2]^{<\omega}$  such that, for every  $y \in Y_1$

$$f(B_{Y_1}(y, F)) \subseteq B_{Y_2}(f(y), H).$$

If  $f$  is a bijection such that  $f$  and  $f^{-1}$  are  $\prec$ -mappings, we say that  $f$  is an *asymorphism*. The subset subsets  $Y_1$  and  $Y_2$  are *coarsely equivalent* if there exist asymorphic subsets  $Z_1 \subseteq Y_1, Z_2 \subseteq Y_2$  such that  $Y_1 = B_{Y_1}(Z_1, F), Y_2 = B_{Y_2}(Z_2, H)$  for some  $F \in [G_1]^{<\omega}, H \in [G_2]^{<\omega}$ . We say that a property  $\mathcal{P}$  of subsets of  $G$ -spaces is *coarse* if  $\mathcal{P}$  is stable under coarse equivalent, and note that "sparse" and "scattered" are coarse properties.

In asymptology, the group  $\oplus_\omega \mathbb{Z}_2$  is known under name the Cantor macrocube, for its coarse characterization see [5].

**Theorem 5.3.** *A subset  $A$  of a  $G$ -space  $X$  is sparse if and only if  $A$  has no subsets asymptotic to the subset  $W_2 = \{g \in \oplus_\omega \mathbb{Z}_2 : \text{supt} g \leq 2\}$  of the Cantor macrocube.*

*Proof.* Apply arguments from [14, Proof of Theorem 3].  $\square$

**Theorem 5.4.** *For a subset  $A$  of a  $G$ -space  $X$ , the following statements are equivalent*

- (i)  $A$  is scattered;
- (ii)  $\Delta_p(A)$  is discrete for each  $p \in X^*$ ;
- (iii)  $A$  contains no piecewise shifted FP-sets;
- (iv)  $A$  contains no subsets coarsely equivalent to the Cantor macrocube.

*Proof.* The equivalence (ii)  $\Rightarrow$  (iii) follows from Theorem 4.4. To prove (i)  $\Rightarrow$  (iii), repeat the arguments from [3, Proof of Theorem 1].

(ii)  $\Rightarrow$  (i). Let  $Y$  be an infinite subset of  $A$ . We denote by  $\mathcal{F}$  the family of all closed invariant subsets of  $X^*$  and put  $\mathcal{F}_Y = \{F \cap Y^* : F \in \mathcal{F}\}$ . By the Zorn's lemma, there exists minimal by inclusion element  $M \in \mathcal{F}_Y$ . We take an arbitrary  $p \in M$  and show that  $\Delta_p(Y)$  is finite. Assume the contrary. Then the set  $\Delta_p(Y)$  has a limit point  $q$ . Since  $M$  is minimal and  $p \in M$ , there exists  $r \in \beta G$  such that  $p = rq$ . By the definition of the action of  $\beta G$  on  $\beta X$ , for every  $P \in p$ , there exists  $Q \in q$  and  $g \in G$  such that  $gQ \subseteq P$ . It follows that  $p$  is a limit point of  $\Delta_p(Y)$ . Hence,  $\Delta_p(Y)$  is not discrete and we get a contradiction.

The implication (i)  $\Rightarrow$  (iv) is evident because the Cantor macrocube is not scattered. To prove (iv)  $\Rightarrow$  (i), we use the characterization of the Cantor macrocube from [5] and the arguments from [3, Proof of the Proposition 3].  $\square$

**Remark 5.5.** Let  $G$  be an amenable group,  $A$  be scattered subset of  $G$ . By [3, Theorem 2],  $\mu(A) = 0$  for each left invariant Banach measure  $\mu$  on  $G$ . This statement cannot be extended to all  $G$ -spaces. As a counterexample, we take  $X = \omega$  and  $G$  is a group of all permutations of  $X$  with finite supports. In this case, each subset of  $X$  is scattered.

Let  $X$  be a  $G$ -space,  $J$  be a translation invariant ideal in  $\mathcal{P}_X$ . We say that a subset  $A$  of  $X$  is

- $J$ -sparse if  $\Delta_p(A)$  is finite for each  $p \in \check{J}$ ;
- $J$ -scattered if, for every subset  $Y$  of  $A$ ,  $Y \notin \check{J}$ , there is  $p \in \check{J} \cap Y^*$  such that  $\Delta_p(Y)$  is finite.

In this context, sparse and scattered subsets coincide with  $[X]^{<\omega}$ -sparse and  $[X]^{<\omega}$ -scattered subsets respectively.

The arguments proving (ii)  $\Rightarrow$  (i) in Theorem 5.4 witness that  $A$  is scattered provided that each point  $p \in \check{J} \cap A^*$  is isolated in  $X^*$ .

**Question 5.6.** *Assume that  $A$  is  $J$ -scattered. Is every point  $p \in \check{J} \cap A^*$  isolated in  $X^*$ ?*

If a subset  $A$  of  $X$  has a subset  $Y \notin J$  coarsely equivalent to  $\oplus_\omega \mathbb{Z}_2$  then  $A$  is not  $J$ -scattered.

**Question 5.7.** *Assume that a subset  $A$  of  $X$  has no subsets  $Y \notin J$  coarsely equivalent to  $\oplus_\omega \mathbb{Z}_2$ . Is  $A$   $J$ -scattered?*

We note that the families  $\sigma(J)$  and  $\partial(J)$  of all  $J$ -sparse and  $J$ -scattered subsets of  $X$  are translation invariant ideals in  $\mathcal{P}_X$  and say that  $J$  is  $\sigma$ -complete (resp.  $\partial$ -complete) if  $\sigma J = J$  (resp.  $\partial(J) = J$ ). We denote by  $\sigma^*(J)$  (resp.  $\partial^*(J)$ ) the intersection of all  $\sigma$ -complete (resp.  $\partial$ -complete) ideals containing  $J$ . Clearly,  $\sigma^*(J)$  and  $\partial^*(J)$  are the smallest  $\sigma$ -complete and  $\partial$ -complete ideals such that  $J \subseteq \sigma^*(J)$  and  $J \subseteq \partial^*(J)$ . We say that  $\sigma^*(J)$  and  $\partial^*(J)$  are the  $\sigma$ -completion and  $\partial$ -completion of  $J$  respectively.

We define a sequence  $(\sigma^n(J))_{n < \omega}$  by the recursion:  $\sigma^0(J) = J$ ,  $\sigma^{n+1}(J) = \sigma(\sigma^n(J))$ , and note that  $\bigcup_{n \in \omega} \sigma^n(J) \subseteq \sigma^*(J)$ . If  $X$  is left regular, by [10, Theorem 4(1)],  $\sigma^*(J) = \bigcup_{n \in \omega} \sigma^n(J)$  and by [10, Theorem 4(2)],  $\sigma^{n+1}([G]^{<\omega}) \neq \sigma^n([G]^{<\omega})$  for each  $n \in \omega$ .

**Question 5.8.** *Is  $\sigma^*(J) = \bigcup_{n \in \omega} \sigma^n(J)$  for each translation invariant ideal  $J$  in an arbitrary  $G$ -space  $X$ ?*

In contrast to  $\sigma$ -completion, for each translation invariant ideal  $J$  in  $\mathcal{P}_X$ , we have  $\partial^*(J) = \partial(J)$ . In particular the ideal  $\partial([X]^{<\omega})$  of all sparse subsets of  $X$  is  $\partial$ -complete. Indeed, assume that  $A \notin \partial(J)$  and choose  $Y \subseteq A$ ,  $Y \notin J$  such that  $\Delta_p(Y)$  is infinite for each  $p \in \check{J} \cap Y^*$ . Then  $Y \notin \partial(Y)$  and  $A \notin \partial^2(J)$ . Hence,  $\partial^2(J) = \partial(J)$  so  $\partial^*(J) = \partial(J)$ .

## REFERENCES

- [1] T. Banach, N. Lyaskovska, *Completeness of translation-invariant ideals in groups*, Ukr. Math. J. **62** (2010), 1022-1031.
- [2] T. Banach, I. Protasov, S. Slobodianiuk, *Densities, submeasures and partitions of groups*, preprint (<http://arxiv.org/abs/1303.4612>).
- [3] T. Banach, I. Protasov, S. Slobodianiuk, *Scattered subsets of groups* preprint (<http://arxiv.org/abs/1312.6946>).
- [4] T. Banach, O. Ravsky, S. Slobodianiuk, *On partitions of  $G$ -spaces and  $G$ -lattices*, preprint (<http://arxiv.org/abs/1303.1427>).
- [5] T. Banach, I. Zarichnyi, *Characterizing the Cantor bi-cube in asymptotic categories*, Groups, Geometry and Dynamics **5** (2011), 691-728.
- [6] J. Erde, *A note on combinatorial derivation*, preprint (<http://arxiv.org/abs/1210.7622>).
- [7] M. Filali, Ie. Lutsenko, I. Protasov, *Boolean group ideals and the ideal structure of  $\beta G$* , Math. Stud. **30** (2008) 1-10.
- [8] N. Hindman, D. Strauss *Algebra in the Stone-Ćech compactification*, 2nd edition, de Gruyter, 2012.
- [9] Ie. Lutsenko, I.V. Protasov, *Sparse, thin and other subsets of groups*, Intern. J. Algebra Computation **19** (2009) 491-510.
- [10] Ie. Lutsenko, I.V. Protasov, *Relatively thin and sparse subsets of groups*, Ukr. Math. J. **63** (2011), 216-225.
- [11] I.V. Protasov, *Selective survey on Subset Combinatorics of Groups*, Ukr. Math. Bull. **7** (2011), 220-257.
- [12] I.V. Protasov, *The Combinatorial Derivation*, Appl. Gen. Topology **14**, 2 (2013), 171-178.
- [13] I.V. Protasov, *The combinatorial derivation and its inverse mapping*, Central Europ. Math. J. **11** (2013), 2176-2181.
- [14] I.V. Protasov, *Sparse and thin metric spaces*, Math. Stud. (to appear).
- [15] I.V. Protasov, S. Slobodianiuk, *Ultracompanions of subsets of groups*, Comment. Math. Univ. Carolin. (to appear), preprint (<http://arxiv.org/abs/1308.1497>).
- [16] I. Protasov, M. Zarichnyi, *General Asymptology*, Math. Stud. Monogr. Ser., Vol. 12, VNTL, Lviv, 2007.